Witten solution of the Gelfand-Dikii hierarchy

S.M.Natanzon

Moscow State University, e-mail: natanzon@mccme.ru

Abstract. We produce formulas, permiting to find the coefficients of Taylor–series expanded of some important solution of the Gelfand-Dikii hierarchy. By the Witten conjecture these coefficients are numbers of intersection of Mumford-Morita-Muller stable cohomological classes of moduli space of n-spin bundles on Riemann surfaces with punctures.

Mathematics Subject Classifications (1991). 14, 35.

Key words: Gelfand–Dikii hierarchy, KP–hierarchy, moduly space, Witten conjecture.

1. Introduction

The n-Gelfand-Dikii hierarchy (n-K∂V hierarchy) usually is described in term of formal pseudo-differential operators. Consider functions $u_j = u_j(x)$ ($j = 0, \ldots, n-2$) of infinite number of variable $x = (x_1, x_2, \ldots.)$. Let us $\partial_i = \frac{\partial}{\partial x_i}$, $\quad \partial = \partial_1$ and $L = \partial^n + u_{n-2}\partial^{n-2} + \cdots + u_1\partial + u_0$. Denote by $L^{\frac{i}{n}}$ the pseudo-differential operator such that $(L^{\frac{i}{n}})^n = L^i$. Let $[L^{\frac{i}{n}}]_+$ be its differential part. Then the n-Gelfand-Dikii hierarchy [GD] is the system of differential equations on u_i, which follow from infinite system of equations

$$\partial_i L = [[L^{\frac{i}{n}}]_+, L] \quad (i = 1, 2, ...).$$

Any solution $(u_0, \ldots, u_{n-2})$ is generated by a function $v(x_1, x_2, \ldots)$. Among these solutions there exists a remarkable solution W, which satisfies the string equation, generates a vacuum vector of W–algebra and has a representation in a form of matrix integral [AM]. We call it Witten's solution because according to the Witten conjecture the

function $F(x_1, x_2, x_3, \dots) = W(x_1, \frac{x_2}{2}, \frac{x_3}{3}, \dots)$ is the generating function for numbers of intersection of Mumford-Morita-Muller stable cohomological classes of moduli space of n-spin bundles on Riemann surfaces with punctures [W]. This conjecture was proved by Kontsevich for $n = 2$ [Ko] and by Witten himself for surfaces of genus 0 [W].

In this paper we find recurrence relation between coefficients of Taylor series of W. This reduces the Witten conjecture to conjecture that n-spin Mamford-Morita-Muller numbers satisfy to the same relations. These relations give also an algorithm for calculation of n-spin Mamford-Morita-Muller numbers in assuming that the Witten conjecture is true. Moreover we prove that $F(x_1, x_2, \dots, x_{n-1}, 0, 0, \dots) = W(x_1, \frac{x_2}{2}, \dots, \frac{x_n}{n-1}, 0, 0, \dots)$ and thus the Witten conjecture is true for numbers $(c_D(\nu), \bar{M}'_{g,s})$ in notation [W].

We prove also that the solution W has a representation $W = \sum\limits_{g=0}^{\infty} W^g$, where W^g are quasihomohenious series of degrees $(1-g)(2+\frac{2}{n})$. This is some indirect corroboration of the Witten conjecture because according to [W] the function F has a representation $F = \sum\limits_{g=0}^{\infty} F^g$, where F^g are quasihomohenious series of degrees $(1-g)(2+\frac{2}{n})$, corresponding to surfaces of genius g.

We prove that $W^0(x_1, \dots, x_{n-1}, 0, \dots) = W(x_1, \dots, x_{n-1}, 0, \dots)$ and therefore $F^0(x_1, \dots, x_{n-1}, 0, \dots) = F(x_1, \dots, x_{n-1}, 0, \dots)$. Last function is polynomial solution of WDVV equations. Some simple formulas for calculation of this solution was found in [N2].

Organisation of the paper is as follow. In sect 2-4 we following by [DN, N1] represent KP hierarchy as a system of differential equation for $v = -\ln\tau$. In section 5 we prove that the n-hierarchy of Gelfand-Dikii is equivalent of a system of differential equations in a form

$$\partial_{i_1} \cdots \partial_{i_k} v = \sum_{m=1}^{\infty} N^m_{i_1 \cdots i_k} \begin{pmatrix} s_1 & \dots & s_m \\ t_1 & \dots & t_m \end{pmatrix} \partial_{s_1} \partial_1^{t_1} v \cdots \partial_{s_m} \partial_1^{t_m} v, \tag{1.1}$$

where $i_j, t_i \geqslant 1, s_i < n$ and $N^m_{i_1 \cdots i_k} \begin{pmatrix} s_1 & \dots & s_m \\ t_1 & \dots & t_m \end{pmatrix}$ are rational constants.

In section 6 we investigate the Witten solution W of the system (1.1).

The author thanks B.A.Dubrovin for fruitful discussions.

2. Combinatorial lemma

For natural $s, i_1, ..., i_n$ and integer not negative $j_1, ..., j_n$ defind $P_s\begin{pmatrix} i_1 & \dots & i_n \\ j_1 & \dots & j_n \end{pmatrix}$ by recurence formulas:

$$1)P_s\begin{pmatrix} i_1 & \dots & i_n \\ 0 & \dots & 0 \end{pmatrix} = 0; \quad 2)P_s\begin{pmatrix} i \\ j \end{pmatrix} = C_s^j \quad \text{for } j > 0;$$

$$3)P_s\begin{pmatrix} i_1 & \dots & i_n \\ j_1 & \dots & j_n \end{pmatrix} = \frac{1}{n!}C_s^{j_1+\dots+j_n}\frac{(j_1+\dots+j_n)!}{j_1!\cdots j_n!} -$$

$$\sum_{q=1}^{n-1} P_s\begin{pmatrix} i_1 & \dots & i_q \\ j_1 & \dots & j_q \end{pmatrix}\frac{1}{(n-q)!}C_{s-(i_1+\dots+i_q+j_1+\dots+j_q)}^{j_{q+1}+\dots+j_n}\frac{(j_{q+1}+\dots+j_n)!}{j_{q+1}!\cdots j_n!}$$

for $(j_1, ..., j_n) \neq (0, ..., 0)$.

Let $\begin{bmatrix} i_1 & \dots & i_n \\ j_1 & \dots & j_n \end{bmatrix}$ be the set of all matrices, which appear from $\begin{pmatrix} i_1 & \dots & i_n \\ j_1 & \dots & j_n \end{pmatrix}$ by permutation of columns. Let $\begin{Vmatrix} i_1 & \dots & i_n \\ j_1 & \dots & j_n \end{Vmatrix}$ be the number of such matrices. Put us

$$P_s\begin{bmatrix} i_1 & \dots & i_n \\ j_1 & \dots & j_n \end{bmatrix} = \sum P_s\begin{pmatrix} a_1 & \dots & a_n \\ b_1 & \dots & b_n \end{pmatrix},$$

where the sum is taken by all

$$\begin{pmatrix} a_1 & \dots & a_n \\ b_1 & \dots & b_n \end{pmatrix} \in \begin{bmatrix} i_1 & \dots & i_n \\ j_1 & \dots & j_n \end{bmatrix}$$

Lemma 2.1. *Let $m > 0, k > 0$ and $j_n \geqslant 1$ for $n \leqslant m$. Then*

$$P_s\begin{bmatrix} i_1 & \dots & i_m & i_{m+1} & \dots & i_{m+k} \\ j_1 & \dots & j_m & 0 & \dots & 0 \end{bmatrix} =$$

$$= \begin{cases} 0, & \text{if } s \geqslant i_1 + ... + i_m + j_1 + ... + j_m, \\ \frac{1}{k!}\begin{Vmatrix} i_{m+1} & \dots & i_{m+k} \\ 0 & \dots & 0 \end{Vmatrix} P_s\begin{bmatrix} i_1 & \dots & i_m \\ j_1 & \dots & j_m \end{bmatrix}, & \text{if } s < i_1 + ... + i_m + j_1 + ... + j_m. \end{cases}$$

Proof: Prove at first the lemma for $m = 1$ using induction by k. For $m = k = 1$

$$P_s \begin{bmatrix} i_1 & i_2 \\ j_1 & 0 \end{bmatrix} = P_s \begin{pmatrix} i_1 & i_2 \\ j_1 & 0 \end{pmatrix} + P_s \begin{pmatrix} i_2 & i_1 \\ 0 & j_1 \end{pmatrix} =$$

$$= \frac{1}{2} C_s^{j_1} - P_s \begin{pmatrix} i_1 \\ j_1 \end{pmatrix} \cdot C^0_{s-(i_1+j_1)} + \frac{1}{2} C_s^{j_1} - P_s \begin{pmatrix} i_2 \\ 0 \end{pmatrix} \cdot C^0_{s-i_2} = C_s^{j_1} - C_s^{j_1} C^0_{s-(i_1+j_1)} =$$

$$= \begin{cases} 0, & \text{if } s \geqslant i_1 + j_1, \\ C_s^{j_1} = P_s \begin{bmatrix} i_1 \\ j_1 \end{bmatrix}, & \text{if } s < i_1 + j_1. \end{cases}$$

Prove the lemma for $m = 1, k = N$, considering that it is proved for $m = 1, k < N$. If $s \geqslant i_1 + j_1$, then

$$P_1 \begin{bmatrix} i_1 & i_2 & \dots & i_{k+1} \\ j_1 & 0 & \dots & 0 \end{bmatrix} =$$

$$= \frac{1}{k!} C_s^{j_1} \left\| \begin{matrix} i_2 & \dots & i_{k+1} \\ 0 & \dots & 0 \end{matrix} \right\| - P_s \begin{pmatrix} i_1 \\ j_1 \end{pmatrix} \cdot C^0_{s-(i_1+j_1)} \cdot \frac{1}{k!} \left\| \begin{matrix} i_2 & \dots & i_{k+1} \\ 0 & \dots & 0 \end{matrix} \right\| = 0.$$

If $s < i_1 + j_1$ than

$$P_1 \begin{bmatrix} i_1 & i_2 & \dots & i_{k+1} \\ j_1 & 0 & \dots & 0 \end{bmatrix} = \frac{1}{k!} C_s^{j_1} \left\| \begin{matrix} i_2 & \dots & i_{k+1} \\ 0 & \dots & 0 \end{matrix} \right\| - A \cdot C^0_{s-(i_1+j_1)} =$$

$$= \frac{1}{k!} \left\| \begin{matrix} i_2 & \dots & , i_{k+1} \\ 0 & \dots & 0 \end{matrix} \right\| P_s \begin{bmatrix} i_1 \\ j_1 \end{bmatrix}.$$

Thus the lemma is proved for $m = 1$.

Prove the lemma for $m = N$ considering that it is proved for $m < N$. Then

$$P_s \begin{bmatrix} i_1 & \dots & i_m & i_{m+1} & \dots & i_{m+k} \\ j_1 & \dots & j_m & 0 & \dots & 0 \end{bmatrix} =$$

$$= \sum_{\begin{pmatrix} \alpha_1 & \dots & \alpha_m \\ \beta_1 & \dots & \beta_m \end{pmatrix} \in \begin{bmatrix} i_1 & \dots & i_m \\ j_1 & \dots & j_m \end{bmatrix}} \Big(\frac{1}{(m+k)!} C_s^{\beta_1 + \dots + \beta_m} \frac{(\beta_1 + \dots + \beta_m)!}{\beta_1! \cdots \beta_m!} C^k_{m+k} \cdot$$

$$\cdot \left\| \begin{matrix} i_{m+1} & \dots & i_{m+k} \\ 0 & \dots & 0 \end{matrix} \right\| -$$

$$\sum_{q=1}^{m} P_s\begin{pmatrix}\alpha_1 & \dots & \alpha_q\\ \beta_1 & \dots & \beta_q\end{pmatrix}\frac{1}{(m+k-q)!}C_{s-(\alpha_1+\dots+\alpha_q+\beta_1+\dots+\beta_q)}^{\beta_{q+1}+\dots+\beta_m}\frac{(\beta_{q+1}+\dots+\beta_m)!}{\beta_{q+1}!\cdots\beta_m!}C_{m+k-q}^{k}\cdot$$

$$\cdot\left\|\begin{matrix}i_{m+1} & \dots & i_{m+k}\\ 0 & \dots & 0\end{matrix}\right\|\Bigg) = P_s\begin{bmatrix}i_1 & \dots & i_m\\ j_1 & \dots & j_m\end{bmatrix}\frac{1}{k!}\left\|\begin{matrix}i_{m+1} & \dots & i_{m+k}\\ 0 & \dots & 0\end{matrix}\right\| -$$

$$-P_s\begin{bmatrix}i_1 & \dots & i_m\\ j_1 & \dots & j_m\end{bmatrix}C^0_{s-(i_1+\dots+i_m+j_1+\dots+j_m)}\frac{1}{k!}\left\|\begin{matrix}i_{m+1} & \dots & i_{m+k}\\ 0 & \dots & 0\end{matrix}\right\| =$$

$$= \begin{cases} 0, & \text{if } s \geqslant i_1+\dots+i_m+j_1+\dots+j_m,\\ P_s\begin{bmatrix}i_1 & \dots & i_m\\ j_1 & \dots & j_m\end{bmatrix}\frac{1}{k!}\left\|\begin{matrix}i_{m+1} & \dots & i_{m+k}\\ 0 & \dots & 0\end{matrix}\right\|, & \text{if } s < i_1+\dots+i_m+j_1+\dots+j_m. \square\end{cases}$$

3. Equations for the Bacher-Akhiezer function

Consider the KP hierarchy. This is a condition of compatibility of the infinite system of the differential equations

$$\frac{\partial\psi}{\partial x_n} = L_n\psi, \tag{3.1}$$

where

$$L_n = \frac{\partial^n}{\partial x_1^n} + \sum_{i=2}^{n} B_n^i(x)\frac{\partial^{n-i}}{\partial^{n-i}x_1},$$

and ψ is a function of type

$$\psi(x,k) = \exp(\sum_{j=1}^{\infty} x_j k^j)(1+\sum_{i=1}^{\infty}\xi_i k^{-i}),$$

(here $k \in \mathbb{C}$ belong to some neighbourhood of ∞ and $x = (x_1, x_2, ...)$ — is a finite sequence).

Put us

$$\partial_i = \frac{\partial}{\partial x_i},\ \partial = \partial_1.$$

A direct calculation gives

Lemma 3.1. *Conditions of compatibility of (1) are*

$$B_s^t = -\sum_{i=1}^{t-1} C_s^i\partial^i\xi_{t-i} - \sum_{j=2}^{t-1} B_s^j\sum_{i=0}^{t-j-1} C_{s-j}^i\partial^i\xi_{t-i-j}, \tag{3.2}$$

$$\partial_n \xi_i = \sum_{j=1}^{n+i-1} C_n^j \partial^j \xi_{i+n-j} + \sum_{k=2}^{n} B_n^k \sum_{j=0}^{n-k} C_{n-k}^j \partial^j \xi_{i+n-j-k}. \quad \square \tag{3.3}$$

In this case ψ is called a *Bacher-Akhiezer function.*

Consider now the function

$$\ln \psi(x,k) = \sum_{j=1}^{\infty} x_j k^j + \sum_{j=1}^{\infty} \eta_j k^{-j},$$

where

$$\xi_j = \sum_{n=1}^{\infty} \frac{1}{n!} \sum_{i_1+\cdots+i_n=j} \eta_{i_1} \cdots \eta_{i_n}.$$

Lemma 3.2. *Let $2 \leqslant t \leqslant s$. Then*

$$B_s^t = -\sum_{n=1}^{\infty} \sum P_s \begin{pmatrix} i_1 & \dots & i_n \\ j_1 & \dots & j_n \end{pmatrix} \partial^{j_1} \eta_{i_1} \cdots \partial^{i_n} \eta_{i_n},$$

where the second sum is taken by all matrices $\begin{pmatrix} i_1 & \dots & i_n \\ j_1 & \dots & j_n \end{pmatrix}$ such that $i_m \geqslant 1, j_m \geqslant 1$ and $i_1 + \cdots + i_n + j_1 + \cdots + j_n = t$.

Proof: An induction by t. For $t = 2$ according to (3.2), $B_s^2 = -s\partial\xi_1 = -P_s\binom{1}{1}\partial\eta_1$. Prove the lemma for $t = N$ considering that it is proved for $t < N$. According to (3.2)

$$B_s^t = -\sum_{i=1}^{t-1} C_s^i \partial^i \Big(\sum_{n=1}^{\infty} \frac{1}{n!} \sum_{i_1+\cdots+i_n=t-i} \eta_{i_1} \cdots \eta_{i_n} \Big) +$$

$$+ \sum_{j=2}^{t-1} \Big(\sum_{n=1}^{\infty} \sum_{i_1+\cdots+j_n=j} P_s \begin{pmatrix} i_1 & \dots & i_n \\ j_1 & \dots & j_n \end{pmatrix} \partial^{j_1} \eta_{i_1} \cdots \partial^{j_n} \eta_{i_n} \Big) \cdot$$

$$\cdot \Big(\sum_{i=0}^{t-j-1} C_{s-j}^i \partial^j \Big(\sum_{n=1}^{\infty} \frac{1}{n!} \sum_{i_1+\cdots+i_n=t-i-j} \eta_{i_1} \cdots \eta_{i_n} \Big) \Big) =$$

$$= -\sum_{n=1}^{\infty} \sum \Big(\frac{1}{n!} C_s^{j_1+\cdots+j_n} \frac{(j_1+\cdots+j_n)!}{j_1! \cdots j_n!} -$$

$$-\sum_{q=1}^{n-1} P_s\begin{pmatrix} i_1 & \dots & i_q \\ j_1 & \dots & j_q \end{pmatrix}\frac{1}{(n-q)!}C_{s-(i_1+\dots i_q+j_1+\dots+j_q)}^{j_{q+1}\cdots j_n}\frac{(j_{q+1}+\cdots j_n)!}{j_{q+1}!\cdots j_n!}\Big)\partial^{j_1}\eta_{i_1}\cdots\partial^{j_n}\eta_{i_n} =$$

$$= -\sum_{n=1}^{\infty}\sum P_s\begin{pmatrix} i_1 & \dots & i_n \\ j_1 & \dots & j_n \end{pmatrix}\partial^{j_1}\eta_{i_1}\cdots\partial^{j_n}\eta_{i_n},$$

where the second sums are taken by all matrices $\begin{pmatrix} i_1 & \dots & i_n \\ j_1 & \dots & j_n \end{pmatrix}$ such that $i_1+\cdots+i_n+j_1+\cdots+j_n = t$, $i_m \geqslant 1, j_m \geqslant 0$. According to lemma 2.1 it is possible to consider that in the last sum $j_m > 0$ for all m. □

Lemma 3.3.

$$\partial_s\eta_r = \sum_{n=1}^{\infty}\sum P_s\begin{pmatrix} i_1 & \dots & i_n \\ j_1 & \dots & j_n \end{pmatrix}\partial^{j_1}\eta_{i_1}\cdots\partial^{j_n}\eta_{i_n},$$

where the second sum is taken by all matrices $\begin{pmatrix} i_1 & \dots & i_n \\ j_1 & \dots & j_n \end{pmatrix}$ *such that* $i_m \geqslant 1, j_m \geqslant 1$ *and* $i_1+\cdots+i_n+j_1+\cdots+j_n = r+s$.

Proof: An induction by r. According to (3.3) and lemma 3.2 for $r = 1$

$$\partial_s\eta_1 = \partial_s\xi_1 = \sum_{j=1}^{\infty}C_s^j\partial^j\xi_{s+1-j} + \sum_{k=2}^{s}B_s^k\sum_{j=0}^{s-k}C_{s-k}^j\partial^j\xi_{1+s-j-k} =$$

$$= \sum_{j=1}^{s}C_s^j\partial^j\xi_{s+1-j} - \sum_{k=2}^{\infty}\Big(\sum_{n=1}^{\infty}\sum_{i_1+\cdots+j_n=k}P_s\begin{pmatrix} i_1 & \dots & i_n \\ j_1 & \dots & j_n \end{pmatrix}\partial^{j_1}\eta_{i_1}\cdots\partial^{j_n}\eta_{i_n}\Big)\cdot$$

$$\cdot\Big(\sum_{j=0}^{s-k}C_{s-k}^j\partial^j\xi_{1+s-j-k}\Big) = \sum_{n=1}^{\infty}\sum P_s\begin{pmatrix} i_1 & \dots & i_n \\ j_1 & \dots & j_n \end{pmatrix}\partial^{j_1}\eta_{i_1}\cdots\partial^{j_n}\eta_{i_n},$$

where the second sum in the last formula is taken by all matrices $\begin{pmatrix} i_1 & \dots & i_n \\ j_1 & \dots & j_n \end{pmatrix}$ such that $i_1+\cdots+i_n+j_1+\cdots+j_n = s+1$, $i_m \geqslant 1$, $j_m \geqslant 0$. According to lemma 1 in this sum it is sufficient consider only matrices, where $j_m > 0$ for all m.

Prove now the lemma for $r = N$, considering that it is proved for $r < N$. According to (3.3)

$$\partial_s\Big(\sum_{n=1}^{\infty}\frac{1}{n!}\sum_{i_1+\cdots+i_n=r}\eta_{i_1}\cdots\eta_{i_n}\Big) = \sum_{j=1}^{s+r-1}C_s^j\partial^j\xi_{s+r-j} + \sum_{k=2}^{\infty}B_s^k\sum_{j=0}^{s-k}C_{s-k}^j\partial^j\xi_{r+s-j-k}.$$

Thus according to lemma 3.2, lemma 2.1 and inductive hypothesis,

$$\partial_s \eta_r = \sum_{j=1}^{\infty} C_s^j \partial^j \Big(\sum_{n=1}^{\infty} \frac{1}{n!} \sum_{i_1+\dots+i_n=s+r-j} \eta_{i_1} \cdots \eta_{i_n} \Big) +$$

$$+ \sum_{k=2}^{\infty} \Big(\sum_{i_1+\dots+j_n=k} P_s \begin{pmatrix} i_1 & \dots & i_n \\ j_1 & \dots & j_n \end{pmatrix} \partial^{j_1} \eta_{i_1} \cdots \partial^{j_n} \eta_{i_n} \Big) \cdot$$

$$\cdot \sum_{j=0}^{s-k} C_{s-k}^j \partial^j \Big(\sum_{n=1}^{\infty} \frac{1}{n!} \sum_{i_1+\dots+i_n=r+s-j-k} \eta_{i_1} \cdots \eta_{i_n} \Big) - \partial_s \Big(\sum_{n=2}^{\infty} \frac{1}{n!} \sum_{i_1+\dots+i_n=r} \eta_{i_1} \cdots \eta_{i_n} \Big) =$$

$$= \sum_{n=1}^{\infty} \sum P_s \begin{pmatrix} i_1 & \dots & i_n \\ j_1 & \dots & j_n \end{pmatrix} \partial^{j_1} \eta_{i_1} \cdots \partial^{j_n} \eta_{i_n},$$

where the second sum in the last formula is taken by all matrices $\begin{pmatrix} i_1 & \dots & i_n \\ j_1 & \dots & j_n \end{pmatrix}$ such that $i_1 + \dots + i_n + j_1 + \dots + j_n = s + r$, $i_m \geqslant 1, j_m \geqslant 1$. $\square$

4. KP hierarchy

According to [DKJM] the Bacher-Akhiezer function ψ is

$$\psi(x,k) = \exp\left(\sum x_j k^j\right) \frac{\tau(x_1 - k^{-1}, x_2 - \frac{1}{2}k^{-2}, x_3 - \frac{1}{3}k^{-3}, \dots)}{\tau(x_1, x_2, x_3, \dots)}$$

for some function $\tau(x_1, x_2, \dots)$. By analogy of [N1] this gives a possibility to describe the KP hierarchy as an infinite system of differential equations on $v(x,k) = -\ln \tau(x,k)$. Really

$$\sum_{j=1}^{\infty} \eta_j k^{-j} = \ln \psi(x,k) - \sum_{j=1}^{\infty} x_j k^j = -v(x_1 - k^{-1}, x_2 - \frac{1}{2}k^{-2}, \dots) + v(x) =$$

$$= \sum_{n=1}^{\infty} \sum_{i_1+\dots+i_n=j} \frac{(-1)^{n+1}}{n! i_1 \dots i_n} \partial_{i_1} \cdots \partial_{i_n} v(x) k^{-j}.$$

Therefore

$$\eta_r = \sum_{n=1}^{\infty} \sum_{i_1+\dots+i_n=r} \frac{(-1)^{n+1}}{n! i_1 \dots i_n} \partial_{i_1} \cdots \partial_{i_n} v. \tag{4.1}$$

Theorem 4.1. *There exist universal rational coefficients*

$$R_r\begin{pmatrix} s_1 & \dots & s_n \\ t_1 & \dots & t_n \end{pmatrix}, R_{ij}\begin{pmatrix} s_1 & \dots & s_n \\ t_1 & \dots & t_n \end{pmatrix}$$

such that

$$\eta_r = \frac{1}{r}\partial_r v + \sum_{n=1}^{\infty}\sum R_r\begin{pmatrix} s_1 & \dots & s_n \\ t_1 & \dots & t_n \end{pmatrix}\partial_{s_1}\partial^{t_1}v\cdots\partial_{s_n}\partial^{t_n}v, \tag{4.2}$$

$$\partial_i\partial_j v = \sum_{n=1}^{\infty}\sum R_{ij}\begin{pmatrix} s_1 & \dots & s_n \\ t_1 & \dots & t_n \end{pmatrix}\partial_{s_1}\partial^{t_1}v\cdots\partial_{s_n}\partial^{t_n}v, \tag{4.3}$$

where the second sums are taken by all matrices $\begin{pmatrix} s_1 & \dots & s_n \\ t_1 & \dots & t_n \end{pmatrix}$ *such that* $s_m, t_m \geqslant 1$, *and the sum* $s_1+\dots+s_n+t_1+\dots+t_n$ *is equal* r *for (4.2) and* $i+j$ *for (4.3).*

Proof: An induction by k and $i+j$. For $i+j=2$ the theorem is obviously. For $r=1$ it follows from (4). Prove the theorem for $i+j=N$ and $r=N-1$, considering that it is proved for $i+j<N$ and $r<N-1$. Later we consider that $s_m, t_m \geqslant 1$ and $\sigma_n = s_1+\dots+s_n+t_1+\dots+t_n$. Then according to (4.1) and (4.3)

$$\eta_r = \frac{1}{r}\partial_r v + \sum_{n=2}^{\infty}\sum_{s_1+\dots+s_n=r}\frac{(-1)^{n+1}}{n!s_1\cdots s_n}\partial_{s_1}\cdots\partial_{s_n}v(x) =$$

$$= \frac{1}{r}\partial_r v + \sum_{n=1}^{\infty}\sum_{\sigma_n=r} R_r\begin{pmatrix} s_1 & \dots & s_n \\ t_1 & \dots & t_n \end{pmatrix}\partial_{s_1}\partial^{t_1}v\cdots\partial_{s_n}\partial^{t_n}v.$$

Thus according to (4.2), (4.3) and lemma 3.3

$$\frac{1}{j}\partial_i\partial_j v = \partial_i\eta_j - \partial_i\Big(\sum_{n=1}^{\infty}\sum_{\sigma_n=j} R_j\begin{pmatrix} s_1 & \dots & s_n \\ t_1 & \dots & t_n \end{pmatrix}\partial_{s_1}\partial^{t_1}v\cdots\partial_{s_n}\partial^{t_n}v\Big) =$$

$$= \sum_{n=1}^{\infty}\sum_{\sigma_n=i+j} P_i\begin{pmatrix} s_1 & \dots & s_n \\ t_1 & \dots & t_n \end{pmatrix}\partial^{t_1}\eta_{s_1}\cdots\partial^{t_n}\eta_{s_n} -$$

$$-\partial_i\Big(\sum_{n=1}^{\infty}\sum_{\sigma_n=j} R_j\begin{pmatrix} s_1 & \dots & s_n \\ t_1 & \dots & t_n \end{pmatrix}\partial^{t_1}\partial_{s_1}v\cdots\partial^{t_n}\partial_{s_n}v\Big) =$$

$$= \sum_{n=1}^{\infty} \sum_{s_1+\cdots+s_n+n=i+j} P_i \begin{pmatrix} s_1 & \dots & s_n \\ 1 & \dots & 1 \end{pmatrix} \partial(\frac{1}{s_1}\partial_{s_1} v) \cdots \partial(\frac{1}{s_n}\partial_{s_n} v) +$$

$$+ \sum_{n=1}^{\infty} \sum_{\sigma_n=i+j, t_1+\cdots+t_n>n} R_{ij} \begin{pmatrix} s_1 & \dots & s_n \\ t_1 & \dots & t_n \end{pmatrix} \partial_{s_1}\partial^{t_1} v \cdots \partial_{s_n}\partial^{t_n} v. \Box$$

Remark 4.1. The system (4.3) was at first deduced in [DN]. The set of its solution bijectively correspond to the set of solutions of (3.1). Up to a constant formal solution of (4.3) is defined by an infinite set of functions of one variable $f_i(x_1) = \partial_i v|_{x_2=x_3=\cdots=0}$ $(i = 1, 2, ...)$.

Remark 4.2. The algorithm described in the proof ot theorem 4.1 gives an algorithm for calculation of all rational constants $R_{ij} \begin{pmatrix} s_1 & \dots & s_m \\ t_1 & \dots & t_m \end{pmatrix}$. The first equations of hierarchy (4.3) are:

$$\partial_2^2 v = \frac{4}{3}\partial_3\partial v - \frac{1}{3}\partial^4 v + 2(\partial^2 v)^2, \tag{4.4}$$

$$\partial_3\partial_2 v = \frac{3}{2}\partial_4\partial v - \frac{3}{2}\partial_2\partial^3 v + 3\partial_2\partial v\partial^2 v,$$

$$\partial_3^2 v = \frac{9}{5}\partial_5\partial v - \partial_3\partial^3 v + \frac{1}{5}\partial^6 v + 3\partial_3\partial v\partial^2 v + \frac{9}{4}(\partial_2\partial v)^2 - 3\partial^4 v\partial^2 v - \frac{9}{4}(\partial^3 v)^2 + 3(\partial^2 v)^3.$$

The equation (4.4) is KP equation twice integrated over x_1.

Theorem 4.2. *If* $\sum_{i=1}^m (t_i + 1) \equiv 1 (mod\ 2)$, *then* $R_{ij} \begin{pmatrix} s_1 & \dots & s_m \\ t_1 & \dots & t_m \end{pmatrix} = 0$.

Proof: The equations of KP–hierarchy are equivalents of equations on function $\tau(x)$ [DKJM]. All these equations can be written simply by means of the "bilinear Hirota operators". We recall the definition of them. If $f(x)$ is a function of one variable, then for any polynomial (or power series) Q the action of the Hirota operator $Q(D_x)f(x) \cdot f(x)$ is defined by

$$Q(D_x)f(x) \cdot f(x) = Q(\partial_y)[f(x+y)f(x-y)]_{y=0}.$$

For functions of several variables the definition is similar. The generating function for the equations of the KP hierarchy has the form

$$\sum_{j=0}^{\infty} p_j(-2y) p_{j+1}(\widetilde{D}) \exp\big(\sum_{i=1}^{\infty} y_i D_i\big) \tau \cdot \tau = 0, \tag{4.5}$$

where $y = (y_1, y_2, \dots,)$ are auxiliary independent variables, $\widetilde{D} = (D_1, 2^{-1}D_2, 3^{-1}D_3, \dots)$, D_j is the Hirota operator in the variable x_j and p_j are the Schur polynomials defined from the following expansion:

$$\exp\big(\sum_{j=1}^{\infty} x_j k^j\big) = \sum_{j=0}^{\infty} k^j p_j(x_1, \dots, x_j).$$

All monomials of odd degree give trivial Hirota operators. Therefore if $\tau(x)$ is a solution of (4.5), then $\widetilde{\tau}(x) = \tau(-x)$ is also solution of (4.5). Moreover, according to [DN] a function τ is a solution of the system (4.5) if and only if $v = -\ln(\tau)$ is a solution of the system (4.3). Thus, $v(x)$ is a formal solution of the system (4.3), if and only if $\widetilde{v}(x) = v(-x)$ is a formal solution of the system (4.3). This is equivalent of the affirmation of theorem 4.2. $\square$.

5. Gelfand–Dikii hierarchy

According to [S] the set of solution of n–Gelfand–Dikii hierarchy bijectively correspond to the set of nondepending from x_n solutions of KP hierarchy. In this case according to theorem 4.1

$$0 = \partial_m\partial_n v = \frac{mn}{m+n-1}\partial_{n+m-1}\partial v + \sum_{m=1}^{\infty}\sum R_{mn}\begin{pmatrix} s_1 & \dots & s_m \\ t_1 & \dots & t_m \end{pmatrix}\partial_{s_1}\partial^{t_1}v\cdots\partial_{s_m}\partial^{t_m}v,$$

where $1 \leqslant s_j \leqslant n+m-2,\ t_j \geqslant 1$. This gives recurrence formulas expressing $\partial_k\partial v$ for $k > n$ via $\partial_r\partial v$ for $r < n$. Thus we have relations

$$\partial\partial_{n+r}v = \sum_{m=1}^{\infty}\sum N^m_{1(n+1)}\begin{pmatrix} s_1 & \dots & s_m \\ t_1 & \dots & t_m \end{pmatrix}\partial_{s_1}\partial^{t_1}v\cdots\partial_{s_m}\partial^{t_m}v, \tag{5.1}$$

where $t_j \geqslant 1, s_j < n, \quad \sum_{j=1}^{m}(s_j + t_j) = n + r + 1$.

Example. For $n = 2$ the system (5.1) passes to $K\partial V$ hierarchy.

Compering the systems (5.1) and (4.3) we find the system

$$\partial_i\partial_j v = \sum_{m=1}^{\infty}\sum N^m_{i_j}\begin{pmatrix} s_1 & \dots & s_m \\ t_1 & \dots & t_m \end{pmatrix}\partial_{s_1}\partial^{t_1}v\cdots\partial_{s_m}\partial^{t_m}v, \tag{5.2}$$

where $i, j \geqslant 1,\ 1 \leqslant s_\alpha \leqslant n-1,\ t_\alpha \geqslant 1,\ \sum_{i=1}^m (s_\alpha + t_\alpha) = i + j$ and $N_{ij}^m \begin{pmatrix} s_1 & \dots & s_m \\ t_1 & \dots & t_m \end{pmatrix}$ are some universal rational coefficients.

Examples.

1. For $n = 3$ the first equation from the system (5.2) is the Boussinesq equation

$$\partial_2^2 v = -\frac{1}{3}\partial^4 v + 2(\partial^2 v)^2.$$

2. For $n = 4$ the first equations of the system (5.2) are

$$\partial_2^2 v = \frac{4}{3}\partial_3 \partial v - \frac{1}{3}\partial^4 v + 2(\partial^2 v)^2,$$

$$\partial_3 \partial_2 v = -\frac{3}{2}\partial_2 \partial^3 v + 3\partial_2 \partial v \partial^2 v,$$

$$\partial_3^2 v = -\frac{1}{4}\partial_3 \partial^3 v + \frac{1}{8}\partial^6 v + \frac{9}{8}(\partial_2 \partial v)^2 - \frac{9}{8}(\partial^3 v)^2 - \frac{9}{4}\partial^4 v \partial^2 v + 3(\partial^2 v)^3. \square$$

Theorem 5.1. *The Gelfand–Dikii hierarchy is equivalent to a system of differential equations in a form*

$$\partial_{i_1} \cdots \partial_{i_k} v = \sum_{m=1} \sum_{i_1 \cdots i_k} N_{i_1 \cdots i_k}^m \begin{pmatrix} s_1 & \dots & s_m \\ t_1 & \dots & t_m \end{pmatrix} \partial_{s_1} \partial^{t_1} v \cdots \partial_{s_m} \partial^{t_m} v, \qquad (5.3)$$

where $k \geqslant 2, t_j \geqslant 1, s_j < n,\ \sum_{j=1}^k i_j = \sum_{j=1}^m (s_j + t_j),\ \sum_{j=1}^m t_j \geqslant m + k - 2\ \ k + m + \sum_{j=1}^m t_j \equiv 0\ (mod2)$.

Proof: For $k = 2$ the equations (5.3) coincide with the equations (5.2). For $k > 2$ the equations (5.3) are received from equations for $k-1$ by differentiation by ∂_{i_k} and replacing $\partial_i \partial_{i_k} v$ by (5.2). This gives monomials $\partial_{s_1} \partial^{t_1} \cdots \partial_{s_m} \partial^{t_m} v$, where the $\sum_{j=1}^k i_j = \sum_{j=1}^m (s_j + t_j)\ \ \sum_{j=1}^m t_j \geqslant m + k - 2$. The condition $k + m + \sum_{j=1}^m t_j \equiv 0(\mathrm{mod}2)$ follows from theorem 4.2. $\square$

Remark 5.1. Structure of the system (5.3) such that its formal solutions are defined up to constant by arbitrary set of $n-1$ series from one variable $f_i(x_1) = \partial_i v|_{x_2 = x_3 = \dots = 0}\quad (i = 1, ..., n-1)$.

Remark 5.2. Coefficients $N^m_{i_1\cdots i_k}\begin{pmatrix} s_1 & \dots & s_m \\ t_1 & \dots & t_m\end{pmatrix}$ are rational constants. The constructions, described in the proofs of theorems 4.1 5.1, give recurrent formulas for its calculation.

6. Witten solution of the Gelfand–Dikii hierarchy

Follow by Witten [W] let us consider the space $M_{g,s}$ of Riemann surfaces of genus g with s punctures. Correspond to any puncture a pair (k_i, m_i), where $1 \leqslant k_i < n, \quad m_i \geqslant 0$. Witten [W] connects with the set $\{(k_i, m_i) | i = 1, ..., s\}$ a number (correlator) $< \prod\limits_{k,m} \tau^{d_{k,m}}_{k,m} >_g$, where $d_{k,m}$ is the number of pairs (k_i, m_i), that equal to (k, m). The number $< \prod\limits_{k,m} t^{d_{k,m}}_{k,m} >_g$ is equals to the value of some class of cohomology on a compactification of a space of n–spin bundles over $P \in M_{g,s}$ [W]. Put us

$$F^g(t_{1,0}, t_{1,1}, ...) = \sum_{d_{k,m}} < \prod_{k,m} \tau^{d_{k,m}}_{k,m} >_g \prod_{k,m} \frac{t^{d_{k,m}}_{k,m}}{d_{k,m}!}.$$

According to the Witten conjecture the serias $F = \sum\limits_{g=0}^{\infty} F^g$ after the change $t_{k,m} \mapsto -(mn + k)x_{mn+k}$ pass to a formal solution v of the system (5.3) satisfying the equation

$$\partial v = \frac{1}{2} \sum_{i+j=n} ijx_i x_j + \sum_{i=1}^{\infty} (i + n)x_{i+n}\partial_i v, \tag{6.1}$$

$$0 = v(0) = \partial_i v(0) \quad (i = 1, 2, ...).$$

The single such solution W we call the Witten solution of the Gelfand–Dikii hierarchy.

Theorem 6.1. *The Witten solution of the Gelfand–Dikii hierarchy is* $W = \sum\limits_{g=0}^{\infty} W^g$, *where*

$$W^g(x_1, x_2, ...) =$$

$$= \sum_{k=2}^{\infty} \sum_{i_1+\cdots+i_k=(n+1)(2g-2+k)} \frac{(n-1)^{2g-2+k}}{k!} N^{2g-2+k}_{i_1\cdots i_k} \begin{pmatrix} n-1 & \dots & n-1 \\ 2 & \dots & 2\end{pmatrix} x_{i_1} \cdots x_{i_k}.$$

Moreover the function W^g is a quasihomogeous series of degree $(1-g)(2+\frac{2}{n})$ by x_i of degrees $1+\frac{1}{n}-\frac{i}{n}$.

Proof: Compatibility of the equations (5.3) (6.1) follows from [AM], where this solution is represented in a form of matrix integral. According to (6.1) $\partial\partial_i W|_{x_2=x_3=\cdots=0} = \delta_{n-1,i}(n-1)x_1$. These conditions and the equations (5.3) uniquely determine all functions $f_{i_1\cdots i_k}(x_1) = \partial_{i_1}\cdots\partial_{i_k} W|_{x_2=x_3=\cdots=0}$. According to theorem 5.1 if $f_{i_1\cdots i_k}(0)\neq 0$, that $(i_1+\cdots+i_k)\equiv 0 \pmod{(n+1)}$ and

$$f_{i_1\cdots i_k}(0) = (n-1)^m N^m_{i_1\cdots i_k}\begin{pmatrix} n-1 & \dots & n-1 \\ 2 & \dots & 2 \end{pmatrix},$$

where $m=\frac{i_1+\cdots+i_k}{n+1}$. From this theorem follow also that $m\geqslant k-2$ $k+m\equiv 0 \pmod 2$. Therefore $m=2g+k-2$, where $g\geqslant 0$ is a natural number. Thus, $W=\sum W^g$, where

$$W^g(x_1,x_2,...) = \sum_{k=2}^{\infty}\sum_{i_1+\cdots+i_k=(n+1)(2g+k-2)}\frac{1}{k!}f_{i_1\cdots i_k}(0)x_{i_1}\cdots x_{i_k} =$$

$$= \sum_{k=2}^{\infty}\sum_{i_1+\cdots+i_k=(n+1)(2g+k-2)}\frac{1}{k!}(n-1)^{2g-2+k}N^{2g-2+k}_{i_1\cdots i_k}\begin{pmatrix} n-1 & \dots & n-1 \\ 2 & \dots & 2 \end{pmatrix}x_{i_1}\cdots x_{i_k}.$$

The quasihomogeneity of the series W^g follows from $i_1+\cdots+i_k=(n+1)(2g-2+k)$. □

Corollary 6.1. *The Witten solution of the Gelfand–Dikii hierarchy has a representation by the sum of quasihomogeneous series W^g of the same degrees that F^g.*

Proof: According to [W] F^g is a quasihomogeneous series of degree $(1-g)(2+\frac{2}{n})$ by $t_{k,m}$ of degree $1+\frac{1}{n}-k-\frac{m}{n}$. □

Theorem 6.2. *The functions W and W^0 coincide on the set $L_0=(x_1,x_2,\dots,x_{n-1}, 0,0,\dots)$.*

Proof: According to (6.1) $\partial^r\partial_\ell W = 0$ on the set L_0 if $\ell < n-1,\ r > 1$, or $\ell = n-1,\ r > 2$. Besides according to (5.1)

$$\partial\partial_{n+\ell}W = \sum_{m=1}^{\infty}\sum N^m_{1(n+\ell)}\begin{pmatrix} s_1 & \dots & s_m \\ t_1 & \dots & t_m \end{pmatrix}\partial_{s_1}\partial^{t_1}W\cdots\partial_{s_m}\partial^{t_m}W,$$

where $\sum_{i=1}^{m}(s_i+t_i) = n+\ell+1$. Thus, if $\ell < n$ and $\partial\partial_{n+\ell}W \neq 0$, that all numbers t_j are less than 3 and among them is not two number more than 1. But if among the numbers $1 \leqslant t_1,\dots,t_m \leqslant 2$ there is exactly one $t_j = 2$ then $N^m_{1(n+\ell)}\begin{pmatrix} s_1 & \dots & s_m \\ t_1 & \dots & t_m \end{pmatrix} = 0$ by theorem 5.1. Thus,

$$\partial\partial_{n+\ell}W = \sum_{m=1}^{\infty}\sum N^m_{1(n+\ell)}\begin{pmatrix} s_1 & \dots & s_m \\ 1 & \dots & 1 \end{pmatrix}\partial_{s_1}\partial W\cdots\partial_{s_m}\partial W.$$

Moreover, according (6.1) $\partial_\ell W = \partial\partial_{n+\ell}W$ and $\partial_s\partial W = s(n-s)x_{n-s}$ on the set L_0 . Thus from the equality $\sum_{i=1}^{m}(s_i+t_i) = n+\ell+1$ follows that $\partial_\ell W|_{L_0}$ is a quasihomogeneous polynomial of degree $2+\frac{2}{n}-(1+\frac{1}{n}-\frac{\ell}{n}) = 1+\frac{1}{n}+\frac{\ell}{n}$. Thus, $W|_{L_0}$ is a quasihomogeneous polynomial of degree $2+\frac{2}{n}$ $W|_{L_0} = W^0|_{L_0}$.□

Corollary 6.2. $F(x_1,x_2,\dots,x_{n-1},0,0,\dots) = W(x_1,\frac{x_2}{2},\frac{x_3}{3},\dots,\frac{x_{n-1}}{n-1},0,0,\dots)$.

Proof: According to [W] $F^0(x_1,x_2,\dots) = W^0(x_1,\frac{x_2}{2},\frac{x_3}{3},\dots)$ and $F(x_1,x_2,\dots,x_{n-1},0,0,\dots) = F^0(x_1,x_2,\dots,x_{n-1},0,0,\dots)$. Thus the theorem 6.2 imply corollary 6.2. □

Remark 6.1. According to [DVV, Kr] the function $W^0|_{L_o}$ is the potential of Frobenius structure on the space of versal deformations of the singularity A_n. By theorem 6.2 we have $W^0|_{L_0} = W|_{L_0}$. A simple algorithm of calculation of this function describes in [N2].

References

[AM] M.Adler, P.van Moerbeke, A matrix integral solution to two–dimensional W_p–gravity. Commun. Math. Phys. 147 (1992), 25-56.

[DKJM] E.Date, M.Kashiwara, M.Jimbo, T.Miwa, Transformation groups for soliton equation, Proceedings of RIMS Symposium on Non-Linear Integrable System. Singapore: World Science Publ. Co., 1983, 39-119.

[DN] B.A.Dubrovin, S.M.Natanzon, Real theta-function solutions of the Kadomtsev–Petviashvili equation. Math. USSR Irvestiya, 32:2 (1989), 269-288.

[DVV] R.Dijkgraaf, E.Verlinde, H.Verlinde, Topological strings in $d < 1$. Nucl. Phys., B 352 (1991), 59.

[GD] I.M.Gelfand, L.A.Dikii, The resolvent and Hamiltonian systems, Funct. Anal. Appl. 11 (2) (1977), 93-105.

[Ko] M.Kontsevich, Intersection theory on the moduli space of curves and the matrix airy function. Commun. Math. Phys. 147 (1992), 1-23.

[Kr] I.Krichever, The dispersionless Lax equations and topological minimal models. Commun. Math. Phys. 143 (1992), 415-429.

[N1] S.M.Natanzon, Real nonsingular finite zone solutions of solution equations. Amer. Math. Soc. Transl. (2). V.170 (1995), 153-183

[N2] S.M.Natanzon, Formulas for A_n and B_n–solutions of WDVV equations, hep-th/9904103.

[S] M.Sato, Soliton equations and universal Grassmann manifold. Math. Lect. Notes Ser., Vol.18, Sophia University, Tokyo (1984).

[W] E.Witten, Algebraic geometry associated with matrix models of two dimensional gravity, Topological models in modern mathematics (Stony Brook, NY, 1991), Publish or Perish, Houston, TX 1993, 235-269.